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A soliton on a vortex filament

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The intrinsic equation governing the curvature κ and the torsion τ of an isolated very thin vortex filament without stretching in an incompressible inviscid fluid is reduced to a non-linear Schrödinger equation

$$\frac{1}{i}\frac{\partial \psi}{\partial t} = \frac{\partial^2 \psi}{\partial s^2} + \tfrac{1}{2}(|\psi|^2 + A)\,\psi,$$

where t is the time, s the length measured along the filament, ψ is the complex variable

$$\psi = \kappa \exp\left(i\int_0^s \tau\, ds\right)$$

and A is a function of t. It is found that this equation yields a solution describing the propagation of a loop or a hump of helical motion along a line vortex, with a constant velocity 2τ. The relation to the system of intrinsic equations derived by Betchov (1965) is discussed.

1. Introduction

Vortex filaments in a perfect fluid are known to preserve their identity and extensive investigations have been made on the two-dimensional motion of a system of vortices. In the three-dimensional case, however, few examples are known even for a single filament owing to its complicated behaviour.

Recently the so-called localized induction equation which describes asymptotically the motion of a very thin vortex filament has been derived by Arms (1962; from a private communication to Hama) and has been used by Hama (1962, 1963) in order to describe the motion of curved filaments of several shapes. For its derivation Batchelor's book (1967) may be consulted.

The essential feature of this method is the approximation of the local motion of the filament by that of a thin cirular vortex with the same curvature and the neglect of slow variation of its coefficient. As long as the interaction between far distant portions along the filament is neglected, this approximation seems to be valid at least qualitatively, as shown numerically by Hama (1962) for a parabola and experimentally by Kambe & Takao (1971) for a distorted vortex ring. On the basis of this approximation Hasimoto (1971) has shown that the shape of a simply rotating plane filament is that of a plane elastic filament, i.e. the elastica.

In order to consider the complicated behaviour of the filament, however, a system of intrinsic equations for the curvature and the torsion of the filament seems to be useful. Betchov (1965) has derived such a system of equations, which may be reduced to those for a fictitious gas with negative pressure accompanied with complicated non-linear dispersive stresses.

In this paper, a simple intrinsic equation for a complex variable with the curvature as its amplitude and the torsion angle as its phase is derived by a simple procedure starting from the fundamental equations of differential geometry. This equation is found to be the kind of non-linear Schrödinger equation which appears in the theories of non-linear optics and plasma physics (Karpman & Krushkal 1969; Taniuti & Yajima 1969; Asano, Taniuti & Yajima 1969). It is shown that this equation admits a solution describing a solitary wave propagating along a line vortex filament, which induces various types of motion of the filament according to the value of the torsion. In an appendix, a deduction of Betchov's intrinsic equation is made from our equation.

2. Fundamental equations

The motion of a very thin isolated vortex filament $\mathbf{X} = \mathbf{X}(s,t)$ of radius ϵ in an incompressible unbounded fluid by its own induction is described asymptotically by

$$\partial \mathbf{X}/\partial t = G\kappa \mathbf{b}, \tag{2.1}$$

where s is the length measured along the filament, t the time, κ the curvature, $\mathbf{b}$ the unit vector in the direction of the binormal and G is the coefficient of local induction.

$$G = (\Gamma/4\pi)\,[\log(1/\epsilon)+O(1)], \tag{2.2}$$

which is proportional to the circulation Γ of the filament and may be regarded as constant if we neglect the slow variation of the logarithm compared with that of its argument. It should be noted that the interaction of order one between far distant portions of the filament is neglected in this approximation and the local motion is approximated by that of a thin circular ring with the same curvature. In this approximation we are obliged to neglect the tangential motion along the filament due to stretching although it is a very important aspect in many cases.

Then a suitable choice of the units of time and length reduces (2.1) to the non-dimensional form

$$\dot{\mathbf{X}} = \kappa \mathbf{b}, \tag{2.3}$$

where a dot denotes $\partial/\partial t$. Equation (2.3) should be supplemented by the equations of differential geometry (the Frenet–Seret formulae)

$$\mathbf{X}' = \mathbf{t}, \quad \mathbf{t}' = \kappa\mathbf{n}, \qquad (2.4), (2.5)$$

$$\mathbf{n}' = \tau\mathbf{b} - \kappa\mathbf{t}, \quad \mathbf{b}' = -\tau\mathbf{n}, \qquad (2.6), (2.7)$$

where a prime denotes $\partial/\partial s$, τ is the torsion and $\mathbf{t}$, $\mathbf{n}$ and $\mathbf{b}$ are a right-handed system of mutually perpendicular unit vectors parallel to the tangent, the principal normal and the binormal respectively.

Combining (2.6) and (2.7) we have

$$(\mathbf{n}+i\mathbf{b})' = -i\tau(\mathbf{n}+i\mathbf{b}) - \kappa\mathbf{t}, \tag{2.8}$$

which suggests the introduction of new variables

$$\mathbf{N} = (\mathbf{n}+i\mathbf{b})\exp\left(i\int_0^s \tau\, ds\right) \tag{2.9}$$

and

$$\psi = \kappa \exp\left(i\int_0^s \tau\, ds\right). \tag{2.10}$$

Then, from (2.8) and (2.5)

$$\mathbf{N}' = -\psi\mathbf{t}, \tag{2.11}$$

and

$$\mathbf{t}' = \operatorname{Re}[\psi\overline{\mathbf{N}}] = \tfrac{1}{2}(\overline{\psi}\mathbf{N}+\psi\overline{\mathbf{N}}), \tag{2.12}$$

where the bar denotes the complex conjugate and Re the real part.

On the other hand, from (2.4), (2.3) and (2.7) we have

$$\dot{\mathbf{t}} = \dot{\mathbf{X}}' = (\kappa\mathbf{b})' = \kappa'\mathbf{b}-\kappa\tau\mathbf{n} = \kappa\operatorname{Re}[(\kappa'/\kappa+i\tau)(\mathbf{b}+i\mathbf{n})], \tag{2.13}$$

i.e.

$$\dot{\mathbf{t}} = \operatorname{Re}[i\psi'\overline{\mathbf{N}}] = \tfrac{1}{2}i(\psi'\overline{\mathbf{N}}-\overline{\psi}'\mathbf{N}), \tag{2.14}$$

where we have made use of (2.9) and (2.10).

It should be noted that orthogonality relations hold between $\mathbf{t}$, $\mathbf{N}$ and $\overline{\mathbf{N}}$:

$$\mathbf{t}.\mathbf{t} = 1, \quad \mathbf{N}.\overline{\mathbf{N}} = 2, \quad \mathbf{N}.\mathbf{N} = 0, \quad \mathbf{N}.\mathbf{t} = 0, \text{ etc.} \tag{2.15}$$

The equation governing the evolution of $\mathbf{N}$ is obtained as follows. By putting

$$\dot{\mathbf{N}} = \alpha\mathbf{N}+\beta\overline{\mathbf{N}}+\gamma\mathbf{t}, \tag{2.16}$$

noting the orthogonality (2.15) and its time derivative and using (2.14) and (2.15), we can determine the coefficients α, β and γ in the following way.

$$\begin{aligned} \alpha+\overline{\alpha} &= \tfrac{1}{2}(\dot{\mathbf{N}}.\overline{\mathbf{N}}+\dot{\overline{\mathbf{N}}}.\mathbf{N}) = \tfrac{1}{2}\partial(\mathbf{N}.\overline{\mathbf{N}})/\partial t = 0, \quad \text{i.e.} \quad \alpha = iR,\\ \beta &= \tfrac{1}{2}\dot{\mathbf{N}}.\mathbf{N} = \tfrac{1}{4}\partial(\mathbf{N}.\mathbf{N})/\partial t = 0,\\ \gamma &= -\mathbf{N}.\dot{\mathbf{t}} = -i\psi', \end{aligned} \tag{2.17}$$

where R is an unknown real function. Thus we have

$$\dot{\mathbf{N}} = i(R\mathbf{N}-\psi'\mathbf{t}). \tag{2.18}$$

The time derivative of (2.11) and the s derivative of (2.18) yield respectively

$$\dot{\mathbf{N}}' = -\dot{\psi}\mathbf{t}-\psi\dot{\mathbf{t}} = -\dot{\psi}\mathbf{t}-\tfrac{1}{2}i\psi(\psi'\overline{\mathbf{N}}-\overline{\psi}'\mathbf{N}) \tag{2.19}$$

and

$$\dot{\mathbf{N}}' = i[R'\mathbf{N}-R\psi\mathbf{t}-\psi''\mathbf{t}-\tfrac{1}{2}\psi'(\overline{\psi}\mathbf{N}+\psi\overline{\mathbf{N}})], \tag{2.20}$$

where we have made use of (2.14), (2.11) and (2.12).

Equating the coefficients of $\mathbf{t}$ and the coefficients $i\mathbf{N}$ (those of $\overline{\mathbf{N}}$ are identical) we have

$$-\dot{\psi} = -i(\psi''+R\psi) \tag{2.21}$$

and

$$\tfrac{1}{2}\psi\overline{\psi}' = R'-\tfrac{1}{2}\psi'\overline{\psi}. \tag{2.22}$$

The comparison of expressions for $\dot{\mathbf{t}}'$ from (2.12) and (2.14) leads only to (2.21). Solving (2.22), we have

$$R = \tfrac{1}{2}(\psi\overline{\psi}+A), \tag{2.23}$$

which reduces (2.21) to

$$\frac{1}{i}\frac{\partial\psi}{\partial t} = \frac{\partial^2\psi}{\partial s^2}+\tfrac{1}{2}(|\psi|^2+A)\,\psi, \tag{2.24}$$

where A is a real function of t which can be eliminated by the introduction of the new variable

$$\Psi = \psi \exp\left[-\tfrac{1}{2}i\int_0^t A(t)\,dt\right]. \tag{2.25}$$

This transformation is nothing but a shift of the origin of integration in (2.9) and (2.10); therefore we may take A in (2.24) to be zero without loss of generality.

Equation (2.24) is the non-linear Schrödinger equation which appears in the theory of non-linear optics and of plasma physics. Hence the results from these cases can be easily transferred to our problem.

3. Solitary wave

As a special case let us look for the solution of (2.24) which describes a solitary wave (soliton) which propagates steadily with a constant velocity c along the filament which is straight at infinity, i.e.

$$\kappa = 0, \quad \text{as} \quad s \to \infty. \tag{3.1}$$

In the wave frame of reference, in which κ and τ are functions of

$$\xi = s - ct, \tag{3.2}$$

i.e.

$$\psi = \kappa(\xi)\exp\left[i\int_0^s \tau(\xi)\,ds\right], \tag{3.3}$$

the real and imaginary parts of (2.24) yield respectively

$$-c\kappa[\tau(\xi)-\tau(-ct)] = \kappa'' - \kappa\tau^2 + \tfrac{1}{2}(\kappa^2 + A)\,\kappa \tag{3.4}$$

and

$$c\kappa' = 2\kappa'\tau + \kappa\tau'. \tag{3.5}$$

Equation (3.5) can be integrated to give

$$(c-2\tau)\,\kappa^2 = 0, \tag{3.6}$$

where we have used (3.1) to determine the integration constant. According to (3.6) we have

$$\tau = \tau_0 = \tfrac{1}{2}c = \text{constant} \tag{3.7}$$

if κ is not identically zero; i.e. the torsion is constant along the filament and the velocity of propagation along the filament is twice the torsion.

Then using (3.1), (3.4) is integrated to give

$$\kappa = 2\nu\,\mathrm{sech}\,\nu\xi, \tag{3.8}$$

provided that A is a constant determined by

$$A = 2(\tau_0^2 - \nu^2). \tag{3.9}$$

The actual shape of the filament is determined by substituting (3.7) and (3.8) into (2.4)–(2.7). For this purpose, it is convenient to solve the equation for $\mathbf{b}$ obtained by the substitution of $\mathbf{n}$ and $\mathbf{t}$ from (2.7) and (2.6) into (2.5):

$$\tau_0(\mathbf{t}' - \kappa\mathbf{n}) = [(1/\kappa)(\mathbf{b}'' + \tau_0^2\mathbf{b})]' + \kappa\mathbf{b}' = 0, \tag{3.10}$$

i.e.
$$\frac{d^3}{d\eta^3}\mathbf{b}+\tanh\eta\frac{d^2}{d\eta^2}\mathbf{b}+(T^2+\operatorname{sech}^2\eta)\frac{d}{d\eta}\mathbf{b}+T^2\tanh\eta\mathbf{b}=0, \tag{3.11}$$
where
$$\eta=\nu\xi \quad\text{and}\quad T=\tau_0/\nu. \tag{3.12}$$
The solution of this equation can be easily obtained by noting that
$$\mathbf{B}=d\mathbf{b}/d\eta+\tanh\eta\mathbf{b} \tag{3.13}$$
is the solution of the equation
$$d^2\mathbf{B}/d\eta^2+(T^2+2\operatorname{sech}^2\eta)\mathbf{B}=0, \tag{3.14}$$
which is satisfied by
$$(\tanh\eta\mp iT)\,e^{\pm iT\eta}. \tag{3.15}$$
As particular solutions, we have
$$\mathbf{b}=\operatorname{sech}\eta,\quad (1-T^2\mp 2iT\tanh\eta)\,e^{\pm iT\eta}. \tag{3.16}$$

Next we substitute (3.16) into (2.7), (2.6) and (2.4), and determine the coefficients so as to satisfy, without loss of generality, the conditions for the filament to be parallel to the x axis of the Cartesian co-ordinates (x,y,z) at infinity: i.e.
$$t_x\to 1\quad\text{as}\quad \eta\to\infty \tag{3.17}$$
and
$$n_y+in_z=-i(b_y+ib_z)=e^{i(\tau_0\xi+\sigma(t))}\quad\text{as}\quad\eta\to\infty, \tag{3.18}$$
the latter condition being suggested by the asymptotic behaviour of the solution of (2.5)–(2.7) and the orthogonality relations between $\mathbf{t}$, $\mathbf{n}$ and $\mathbf{b}$. Here $\sigma(t)$ is a real function of t and the suffices x,y,z denote x, y and z components respectively.

After straightforward calculations we have
$$\left.\begin{aligned}&\mathbf{X}: x=s-(2\mu/\nu)\tanh\eta,\quad y+iz=re^{i\Theta},\\ &\mathbf{t}: t_x=1-2\mu\operatorname{sech}^2\eta,\quad t_y+it_z=-\nu r(\tanh\eta-iT)e^{i\Theta},\\ &\mathbf{n}: n_x=2\mu\operatorname{sech}^2\eta\sinh\eta,\quad n_y+in_z=-[1-2\mu(\tanh\eta-iT)\tanh\eta]\,e^{i\Theta},\\ &\mathbf{b}: b_x=2\mu T\operatorname{sech}\eta,\quad b_y+ib_z=i\mu(1-T^2-2iT\tanh\eta)\,e^{i\Theta},\end{aligned}\right\} \tag{3.19}$$
where
$$\left.\begin{aligned}&\mu=\frac{1}{1+T^2}=\frac{\nu^2}{\nu^2+\tau_0^2},\quad r=\frac{2\mu}{\nu}\operatorname{sech}\eta,\\ &\eta=\nu\xi=\nu(s-2\tau_0t),\quad \Theta=T\eta+\nu^2(1+T^2)\,t=\tau_0s+(\nu^2-\tau_0^2)\,t.\end{aligned}\right\} \tag{3.20}$$
Here the integration constants such as $\sigma(t)$ which are functions of time have been determined so that $\mathbf{X}$ and $\mathbf{b}$ satisfy (2.3) in the rest co-ordinate system.

4. Numerical results and discussions

Figures 1 and 2 show the projections of the filament on the x,y, x,z and z,y planes at an instant of time (say $t=0$). It is seen that the filament is confined on an envelope of radius r (except near the centre $x\sim 0$ if $T<1$) which decreases from its maximum $2\mu/\nu$ at $x=0$ to $2\mu/[\nu\cosh(\nu x\pm 2\mu)]\sim 0$ as $x\to\pm\infty$. Our filament is a spiral surrounding this envelope, being approximated by
$$y+iz=r\exp(i\Theta)\to r\exp[i\tau_0(x\pm 2\mu/\nu)+i(\nu^2-\tau_0^2)\,t]\quad\text{as}\quad x\to\pm\infty.$$

However its behaviour near $x = 0$ depends on the value of T. As long as $T \geqslant 1$, $y+iz$ is a single-valued function of x, since x is a monotonic function of η. If $T = 1$, $dx/d\eta$ is zero at $x = \eta = 0$ and the cusp of the envelope appears at $x = 0$, $r = 2\mu/\nu$, though no singularity exists on the filament.

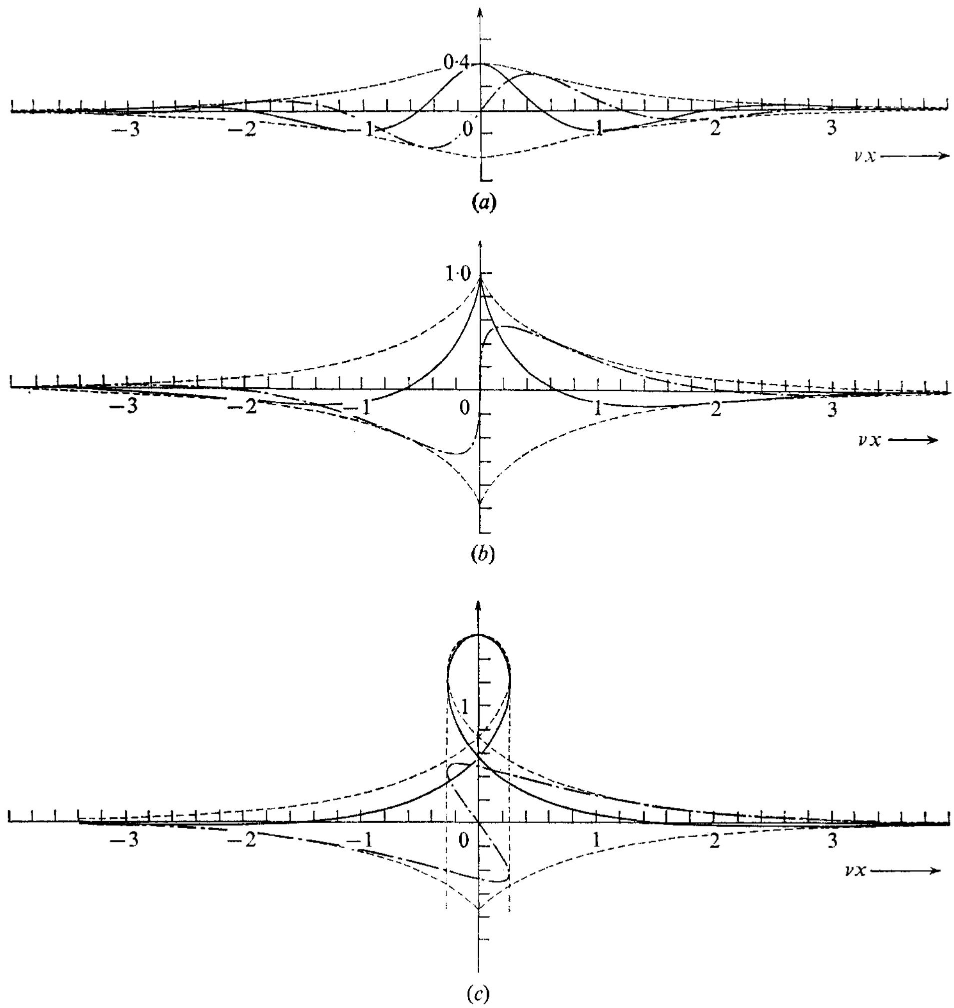

FIGURE 1. ———, projection of filament on the x, y plane with the νy axis vertical; ----, projection on x, z plane with νz axis vertical; ...., envelope with νr axis vertical; the νx axis is horizontal in each case. (a) $T = 2{\cdot}0$, (b) $T = 1{\cdot}0$, (c) $T = 0{\cdot}5$.

For $T < 1$ the filament is twisted and yields a loop in its side view, though no real crossing point exists. As the torsion is decreased the projection to the z, y plane is flattened and in the limit $T \to 0$ the filament is a plane curve with a crossing point which has been noted by Betchov to be unacceptable.

The velocity $\mathbf{v} = \dot{\mathbf{X}} = \kappa \mathbf{b}$ of the filament is obtained from (2.3) and (3.19), also using (3.20) and (3.12), as

$$v_x = 4\mu\tau \operatorname{sech}^2 \eta = \tau_0(\tau_0^2 + \nu^2)\, r^2, \tag{4.1}$$

$$\varpi = v_y + iv_z = \varpi_{\text{rot}} + \varpi_{\text{rad}}, \tag{4.2}$$

where $$\varpi_{\mathrm{rot}} = i\nu^2(1-T^2)(y+iz) = i(\nu^2-\tau_0^2)(y+iz) \tag{4.3}$$

and $$\varpi_{\mathrm{rad}} = 2\nu^2T(y+iz)\tanh\eta = 2\nu\tau_0(y+iz)\tanh\eta. \tag{4.4}$$

It is seen that the motion can be decomposed into three parts: (i) longitudinal motion v_x, (ii) rotation about the x axis ϖ_{rot} and (iii) radial contraction and expansion ϖ_{rad}.

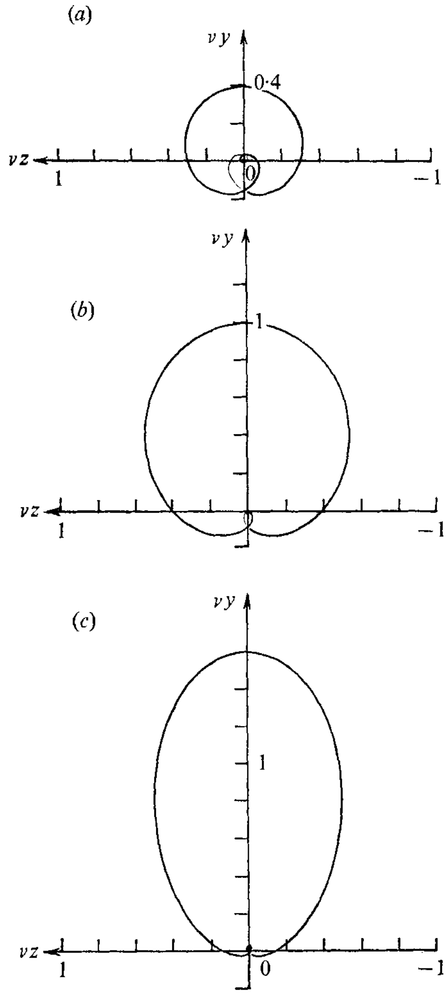

FIGURE 2. Projection of the filament on the z, y plane. (a) $T = 2{\cdot}0$, (b) $T = 1{\cdot}0$, (c) $T = 0{\cdot}5$.

The rotation changes its sign according to that of $T-1$, i.e. A in (3.9). If $T < 1$ the direction of rotation is the same as that of the vorticity at $x = \pm\infty$ and if $T > 1$ it is the opposite. It is interesting to note that no actual rotation occurs if $T = 1$, i.e. $A = 0$. This behaviour may be attributed to the appearance of a loop in the side view for $T < 1$, in contrast to the dominance of the spiral for $T > 1$.

The magnitude of v_x depends on the curvature, which is proportional to r, and the orientation of the looping to the x axis, b_x, which is proportional to τr. The faster motion of the larger looping seems to be coupled with the radial expansion by ϖ_{rad}, leading to the propagation of our solitary wave along the filament. Notice that we have radial expansion or contraction according as $\tau\eta$ is positive or negative. Some of these features are seen to be in accordance with Hama's (1963) numerical experiments on the filament which has Gaussian shape initially.

Though this behaviour of the vortex filament may be temporary, judging from its approximate nature and possible instability, the author hopes that it might be observed in some vortex system such as highly sheared stream or rotating flow. In this connexion, it may be noted that Yajima & Outi (1971) made a numerical calculation on the stability of solitary waves for a non-linear Schrödinger equation and have shown that they are fairly stable.

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Appendix. Relation to Betchov's intrinsic equation

Betchov (1965) has derived a system of intrinsic equations governing essentially two variables

$$\rho = \kappa^2, \quad u = 2\tau. \tag{A 1}$$

In order to derive his equation it is convenient to introduce the potential

$$\Phi = \int_0^s u\,ds = 2\int_0^s \tau\,ds \tag{A 2}$$

into (2.24) and differentiate $\psi = \sqrt{\rho}\exp(\frac{1}{2}i\Phi)$ logarithmically. Comparing the real and imaginary parts of

$$\frac{1}{2i}\left(\frac{\dot\rho}{\rho}+i\dot\Phi\right) = \frac{1}{2}\left(\frac{\rho''}{\rho}-\frac{\rho'^2}{\rho^2}+i\Phi''\right)+\frac{1}{4}\left(\frac{\rho'}{\rho}+i\Phi'\right)^2+\tfrac{1}{2}(\rho+A) \tag{A 3}$$

and differentiating the former with respect to s, we have

$$\dot u + uu' = \rho' + \left(-\frac{\rho'^2}{2\rho^2}+\frac{\rho''}{\rho}\right)', \tag{A 4}$$

$$\dot\rho + u\rho' = -\rho u'. \tag{A 5}$$

These are reduced to the equations given by Betchov (1965, equations (2.16) and (2.23)) if we put $\rho = K$ and $u = 2T$.

It may be noted that (A 5) and (A 4) supplemented by (A 5) yield the conservation forms

$$\frac{\partial\rho}{\partial t}+\frac{\partial}{\partial s}(\rho u) = 0 \tag{A 6}$$

and

$$\frac{\partial}{\partial t}(\rho u)+\frac{\partial}{\partial s}\left[\rho u^2-\tfrac{1}{2}\rho^2-\rho\frac{\partial^2}{\partial s^2}(\log\rho)\right] = 0 \tag{A 7}$$

respectively. By assuming the same dependence on ξ as that in (3.2) and (3.3) we can obtain the same results as in §3.

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